A quantum annealing approach to the Minimum Multicut problem on general graphs

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1 Introduction

In this talk,

◇ A would like to discuss the quantum annealing approach to the solution of combinatorial optimization problems:
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\[ \text{Problem} \rightarrow \text{QUBO} \rightarrow \text{Embedding into the hardware} \]

It is considered the Minimum Multicut problem which is NP-hard on trees and in general graphs.
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◊ We discuss the limitations of the current family of quantum annealing processors.
Contents

Section 2: Quantum annealing

Section 3: Combinatorial optimization

Section 4: Mapping of the Minimum multicut to QUBO

Section 5: Embedding into the hardware

Section 6: Hardware simulation

Section 7: Summary and conclusions
2 Quantum annealing

- QA annealing is used to traverse from the ground state of an initial Hamiltonian to the ground state of the final Hamiltonian. [Finnila et al., 1994] [Kodawaki-Nishimori, 1998] [Farhi et al., 2001]

\[ H(\tau) = A(s)H_I + B(s)H_{\text{problem}}, \]

\[ H_{\text{problem}} = \sum_{i}^{N} h_i \sigma_i^z + \sum_{i < j}^{N} J_{ij} \sigma_i^z \sigma_j^z, \quad H_I = \sum_{i} \sigma_i^x \]

\[ t_f = 20, \ldots, 2000 \mu s \]
Adiabatic evolution

\[ i \frac{d |\Psi(t)\rangle}{dt} = H(t) |\Psi(t)\rangle \]
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Adiabatic Theorem: [BornFock '28, Kato '51]

\[ H(0) \rightarrow H(T) \]

\[ t \gg \min_t \{ \gamma(t) \} \]

\[ \gamma(t) = E_1(t) - E_0(t) \]

Linear interpolation between \( H_0 \) and \( H_1 \):

\[ H(s) = (1 - s) H_0 + s H_1, \quad s = \frac{t}{T} \]

\[ A(s) \sim (1 - s)^{5/2}, \quad B(s) \sim s^{5/20} \]
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Adiabatic Theorem: [BornFock '28, Kato '51]

|\Psi(0)\rangle \text{ Ground state of } H(0) \rightarrow |\Psi(T)\rangle \text{ ground state of } H(T)

\[ T \gg \frac{1}{\min_t \{\gamma(t)\}^2}, \quad \gamma = E_1(t) - E_0(t) \]

No crossing in the paths of the corresponding eigenvectors.
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\[ A(s) \sim (1 - s), \quad B(s) \sim s \]
(Experimental) Quantum annealing

\[ H(\tau) = A(s) \sum_i \sigma_i^x + B(s) H_{\text{problem}} \]

\[ H_{\text{problem}} = \sum_i h_i \sigma_i^z + \sum_{j > i} J_{ij} \sigma_i^z \sigma_j^z \]
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[Lanting et al, 2014]
Adiabatic quantum optimization

- The ground state of $H_p$ corresponds to a configuration $s = (s_1, \ldots, s_N) \in \{+1, -1\}^N$ of spins that minimize the following energy function

$$E(s) = \sum_{i}^{N} h_i s_i + \sum_{j>i}^{N} J_{ij} s_i s_j.$$
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Finding $s^*$ with minimum energy $E(s^*)$ is an NP-hard \(^1\) problem even on planar graphs. [Barahona, 1982]

From classical objective function to quantum Hamiltonian

Find the optimal assignment $s^* = (s_1^*, \ldots, s_N^*)$

$E(s) = \sum_{i} h_i s_i + \sum_{j>i} J_{ij} s_i s_j$

Find the ground state $|\psi_g\rangle = |s^*\rangle = |s_1^*, \ldots, s_N^*\rangle$

$H_p = \sum_{i} h_i \sigma_i^z + \sum_{j>i} J_{ij} \sigma_i^z \sigma_j^z$
3 Combinatorial optimization

- NPO is the class of optimization problems, NP-hard are the most difficult problems in NPO

- Factor $\epsilon$-approximation algorithms $A$ for problem $\Pi$,
  \[ \forall x \in \Pi : \text{cost}_\Pi(x, A(x)) \leq \epsilon \cdot \text{OPT}(x). \]

- APX $\subseteq$ NPO class of problems that can be approximated in polynomial time for some $\epsilon > 1$. 

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The concept of inapproximated problems

Theorem [ALM, 1992]: There is a fixed $\epsilon > 0$ and a polynomial-time reduction $\tau$ from SAT to MAX-3SAT such that for every boolean formula $I$:

$$I \in \text{SAT} \implies \text{MAX-3SAT}(\tau(I)) = 1$$
$$I \notin \text{SAT} \implies \text{MAX-3SAT}(\tau(I)) < \frac{1}{1 + \epsilon}.$$ 

In other words, achieving an approximation ratio $1 + \epsilon$ for MAX-3SAT is NP-hard.
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### Classification of inapproximated problems [Arora-Lund, 1996]

<table>
<thead>
<tr>
<th>Class</th>
<th>Representative problem</th>
<th>Hard ratio</th>
<th>Best ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>MAX-3SAT</td>
<td>$1 + \epsilon$</td>
<td>1.2987 [AHO+97]</td>
</tr>
<tr>
<td></td>
<td>MULTIWAY CUTS</td>
<td></td>
<td>$3/2 - 1/</td>
</tr>
<tr>
<td>II</td>
<td>MINIMUM SETCOVER</td>
<td>$O(\log n)$</td>
<td>$1 + \ln</td>
</tr>
<tr>
<td>III</td>
<td>NEAREST LATTICE VECTOR</td>
<td>$2^n \log^{1-\gamma}$</td>
<td>Not in APX [ABS+97]</td>
</tr>
<tr>
<td>IV</td>
<td>MAXIMUM CLIQUE</td>
<td>$n^\epsilon$</td>
<td>$O\left(\frac{n}{(\log n)^2}\right)$ [BH92]</td>
</tr>
</tbody>
</table>
Minimum multicut: Given a weighted graph $G = (V, E, w)$ and a set of pairs $H = \{(s_1, t_1), \ldots, (s_k, t_k)\} \subset V \times V$, find a multi-cut with minimum capacity, i.e., a subset $E' \subseteq E$ such that the removal of $E'$ from $E$ disconnects $s_i$ from $t_i$ for every pair $(s_i, t_i)$, where the capacity of $E'$ is given as $\sum_{e \in E'} w(e)$. 
4 Mapping of the Minimum multicut to QUBO

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Min s-t cut
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Min s-t cut

3-multicut

Cut

• For $k = 1, 2$, it is solvable in polynomial time. [Bollobas, 79] [Seymour, 79]
• For $k \geq 3$, Minimum Multi-Cut becomes APX-hard. [Dahlhaus, 94]
• It is NP-hard even if restricted to trees of height 1. [Garg et al., 97]
**Minimum multicut**: Given a weighted graph $G = (V, E, w)$ and a set of pairs $H = \{(s_1, t_1), \ldots, (s_k, t_k)\} \subseteq V \times V$, find a multi-cut with minimum capacity, i.e., a subset $E' \subseteq E$ such that the removal of $E'$ from $E$ disconnects $s_i$ from $t_i$ for every pair $(s_i, t_i)$, where the capacity of $E'$ is given as $\sum_{e \in E'} w(e)$.

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QUBO formulation of Minimum multicut in trees

For each edge $e \in G$, $x_e = 1$ (in the cut), 0 (not in the cut)

$$h_G = h_{\text{weight}} + h_{\text{penalty}}$$

1. $h_{\text{weight}} = \sum_{e \in G} w(e)(1 - x_e)$

2. $h_{\text{penalty}} = \lambda_{\text{path}} \sum_{i=1}^{k} \prod_{e \in p_i} x_e$

   $p_i$ is the path from $s_i$ to $t_i$,

   $\lambda_{\text{path}} = \sum_{e \in p_i} w(e)$

3. $\text{deg}(h_{\text{penalty}}) = \max_i \{\text{length}(p_i)\}$

There exists a unique path between every pair of vertices in a tree.
Reduction methods

\[ f(x) = \sum_{S \subseteq [1,n]} a_S \prod_{j \in S} x_j \]

\[ \downarrow \tau_r \]

\[ f(x) = \min_{w \in \{0,1\}^m} g(x,w) \]

\[ \deg\{g(x,w)\} \leq 2 \]

- \( w \) “ancilla variables”
- \( \tau_r \) “polynomial reduction”

(a) Negative terms can be reduced using only one extra ancilla variable
[Freedman-Drineas, 2005]

\[ -x_1 x_2 \cdots x_d = \min_{w \in \{0,1\}} w \left( (d-1) - \sum_{j=1}^{d} x_j \right) \]

(b) For positive terms, only \( \left\lfloor \frac{d-1}{2} \right\rfloor \) new ancilla variables are added.

\[ \prod_{j=1}^{d} x_j = S_2 + \min_{w \in \{0,1\}^k} B - 2AS_1 \]

if \( d = 2k+2, \)

\[ \prod_{j=1}^{d} x_j = S_2 + \min_{w \in \{0,1\}^k} B - 2AS_1 + w_k (S_1 - d+1) \]

if \( d = 2k+1. \)

See [Ishikawa, 2011].

(c) In the penalty approach, for each occurrence of \( xy \), a new term is added.
[Boros-Hammer, 2002]

\[ M(x y - 2x w - 2y w + 3w) \]

Upper bound: \( M = 1 + 2 \sum_{S \subseteq [1,n]} a_S \)

Ancilla variables: \( O(n^2 \log \deg(f)) \)

Bad news: large coefficients
Example of reduction (1)

\[ H = \{(6, 10), (2, 18), (11, 17), (14, 19), (8, 13), (10, 11), (3, 5), (13, 17), (7, 14), (6, 20)\} \]

\[ h_G = 14 - x_1 - x_2 - x_3 - x_4 + 9x_5 - x_6 - x_7 - x_8 - x_9 - x_{10} - x_{11} - x_{12} - x_{13} + 9x_{14} + 10x_1x_2x_3x_4 + 10x_6x_7 + 10x_6x_8x_9 + 10x_2x_3x_4x_5x_{10}x_{11} + 10x_3x_4x_8 + 10x_2x_3x_{12} + 10x_2x_6x_7x_8 + 10x_2x_{12}x_{13} \]

\[ h_G^{\text{qubo}} : 22 \text{ logical variables, 51 physical qubits} \]
Example of reduction (2)

\[
H = \{(6, 10), (2, 18), (11, 17), (14, 19), (8, 13), (10, 11), (3, 5), (13, 17), (7, 14), (6, 20)\}
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h_G = 14 - x_1 - x_2 - x_3 - x_4 + 9x_5 - x_6 - x_7 - x_8 - x_9 - x_{10} - x_{11} - x_{12} - x_{13} + 9x_{14} + 10x_1x_2x_3x_4 + 10x_6x_7 + 10x_6x_8x_9 + 10x_2x_3x_4x_5x_{10}x_{11} + 10x_3x_4x_8 + 10x_2x_3x_{12} + 10x_2x_6x_7x_8 + 10x_2x_12x_{13}
\]

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QUBO formulation of Minimum multicut on general graphs

Given a graph $G = (V, E)$ and a set of pairs $H = \{(s_1, t_1), \ldots, (s_k, t_k)\}$. The Minimum multicut problem can be logically formulated as follows:

$$\min_{C \subseteq E} |C|. \bigwedge_{(s_i, t_i) \in H} \neg \text{connected}(s_i, t_i, C)$$

where

$$\text{connected}(s_i, t_i, C) \equiv \forall U \subseteq V. \varphi(s_i, t_i, C)$$

and

$$\varphi(s_i, t_i, C) \equiv ((s_i \in U \land t_i \notin U) \rightarrow \exists x \in U. \exists y \notin U. \exists e \in E. \text{inc}(x, e) \land \text{inc}(y, e) \land e \notin C).$$

To verify if a given subset $C \subseteq E$ is a cut in $G$ that disconnect every pair $(s_i, t_i)$, then it is sufficient to find a subset $U \subseteq V$ such that $\neg \text{connected}(s_i, t_i, C)$ is true.
**Mapping:** Logical variables $y_{uw}$ and $x^i_v$

- For each $\{u, w\} \in E$, $y_{uw} = 1 \ (0)$ if $\{u, w\}$ is (not) selected for a cut.
- For each $v \in V$ and $i = 1, \ldots, k$, $x^i_v = 1 \ (0)$ if $v$ is (not) in $U$ where $U$ is a subset of $V$.

**Construction:** Let $f_G$ be defined as

$$f_G = \text{card}(y_{uw}) + \alpha \cdot \text{penalty}(x_v, y_{uw}, H)$$

where

$$\text{card}(y_{uw}) = \sum_{\{u, w\} \in E} y_{uw} \quad \text{and}$$

$$\text{penalty} = \sum_{i=1}^k \left( - (x^i_{s_i} \oplus x^i_{t_i}) + \sum_{\{u, w\} \in E} (x^i_u \oplus x^i_w) \oplus y_{uw} \right)$$

$$= \sum_{i=1}^k \left( 1 - x^i_{s_i} - x^i_{t_i} + 2 x^i_{s_i} x^i_{t_i} + \sum_{\{u, w\} \in E} (x^i_u + x^i_w + y_{uw} - 2 x^i_u x^i_w - 2 x^i_u y_{uw} - 2 x^i_w y_{uw} + 4 x^i_u x^i_w y_{uw}) \right)$$
Using the Ishikawa method we obtain

\[
\text{penalty} = \sum_{i=1}^{k} \left( 1 - x_{s_i}^i - x_{t_i}^i + 2x_{s_i}^i x_{t_i}^i + \right.
\]

\[
\sum_{\{u,w\} \in E} \left( x_u^i + x_w^i + y_{uw} - 2x_u^i x_w^i - 2x_u^i y_{uw} - 2x_w^i y_{uw} + 4(x_u^i x_w^i + x_u^i y_{uw} + x_w^i y_{uw} +
\]

\[
z_{uw}^i (1 - x_u^i - x_w^i - y_{uw})) \right)
\]

\[
= \sum_{i=1}^{k} \left( 1 - x_{s_i}^i - x_{t_i}^i + 2x_{s_i}^i x_{t_i}^i + \right.
\]

\[
\sum_{\{u,w\} \in E} \left( x_u^i + x_w^i + y_{uw} + 2x_u^i x_w^i + 2x_u^i y_{uw} + 2x_w^i y_{uw} + 4z_{uw}^i (1 - x_u^i - x_w^i - y_{uw})) \right)
\]

where \(z_{uw}^i\) are ancilla variables.

\(f_G\) uses \(k(n + m) + m\) variables.

\(\alpha\) is upper bounded by \(\text{card}(y_{uw})\).
Example of construction

Boolean variables to represent the given problem:

\[ x_1^1, x_2^1, x_3^1, x_4^1, x_5^1, x_6^1, x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, y_{12}, y_{13}, y_{16}, y_{23}, y_{25}, y_{34}, y_{45}, y_{46}, y_{56} \]

Ancilla variables

\[ z_{12}^1, z_{13}^1, z_{16}^1, z_{23}^1, z_{25}^1, z_{34}^1, z_{45}^1, z_{46}^1, z_{56}^1, z_{12}^2, z_{13}^2, z_{16}^2, z_{23}^2, z_{25}^2, z_{34}^2, z_{45}^2, z_{46}^2, z_{56}^2 \]
5 Summary and conclusions

- The programming model is problem dependent.

- Can we avoid the reduction of pseudo-Boolean functions into QUBO?

- The minimum embedding is not always the best choice.

- Approximate solutions are also useful.

- To investigate programming inapproximated problems.
Thanks for your kind attention!

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